Twisted equivariant K–theory with complex coefficients

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ABSTRACT: Using a global version of the equivariant Chern character, we describe an effective method for computing the complexified twisted equivariant *K*-theory of a space with a compact Lie group action, in terms of fixed-point data. We apply this to the case of a compact group acting on itself by conjugation, and relate the result to the Verlinde algebra of the group, and to the Kac character formula. The Verlinde formula for the dimension of the space of conformal blocks is also discussed in this context.

0. Introduction

Let X be a locally compact topological space acted upon by a compact Lie group G. The equivariant K-theory $K_G^*(X)$ was defined by Atiyah and Segal; some foundational papers are [S] and [AS1]. *Twisted* versions of K-theory, both equivariant and not, have recently attracted some attention. The equivariant twistings we consider are classified, up to isomorphism, by an equivariant cohomology class $(\varepsilon,\tau) \in H_G^1(X;\mathbb{Z}/2)\times H_G^3(X;\mathbb{Z})$. (A twisting is a representative cocycle for such a class, in some model of equivariant cohomology). For torsion, non-equivariant twistings, the relevant K-theory was first introduced in [DK]; subsequent treatments ([R] and, more recently, [BCMMS], [A]), remove this torsion assumption. We recall, for convenience, the topologist's definition. Because the projective unitary group $\mathbb{P}\mathrm{U}$ has classifying space $K(\mathbb{Z};3)$, a class $\tau \in H^3(X)$ defines a principal $\mathbb{P}\mathrm{U}$-bundle over X, up to isomorphism. To such a bundle we associate the Ad-bundle $\mathbb{F}_X$ of Fredholm operators. The negative ${}^{\tau}K(X)$ groups are the homotopy groups of the space of sections of $\mathbb{F}_X$; the others are determined by Bott periodicity. In the presence of a group action, the equivariant groups arise similarly, from invariant sections. In this paper, we shall implicitly assume the basic topological properties of twisted K-theory, whose justification is postponed to future work. Also, we confine ourselves to H^3 twistings in the body of the paper; H^1 twistings are discussed in the Appendix.

One of the basic results [S] of the equivariant theory expresses, in terms of fixed-point data, the localization of $K_G^*(X)$ at prime ideals in the representation ring R_G of G. The situation simplifies considerably after complexification, when the maximal ideals in R_G are the conjugacy classes[4]. Recall that, in the non-equivariant case, the Chern character maps complex K-theory isomorphically onto complex cohomology; the localization results can be assembled into a description of complex equivariant K-theory by a *globalized Chern character* ([AS2], [Ro]), supported over the entire group. Part I of our paper generalizes these results to the twisted case: in §3, we discuss the twisted Chern character, while the main result, Theorem 2.4, describes ${}^{\tau}K_G^*(X;\mathbb{C})$ in terms of (twisted) equivariant cohomology of fixed-point sets, with coefficients in certain equivariantly flat complex line bundles. (For orbifolds, this is Vafa's *discrete torsion* [V], [VW]).

In Part II, we apply our main result to the G-space $X = G$, with G acting by conjugation. For simplicity, we restrict most of the discussion to the case of connected groups with torsion-free π_1. (The nice property shared by such groups is that all centralizers of group elements are connected). For non-singular twistings, the R_G-module ${}^{\tau}K_G^*(X;\mathbb{C})$ is supported at finitely many conjugacy class-

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[4] More precisely, the semi-simple conjugacy classes in the complexified group

es in G (§4). Furthermore, ${}^{\tau}K_G^*(X;\mathbb{C})$ is an algebra under the *Pontryagin product*, and in §6 we shall see that this conjugacy-class decomposition diagonalizes the product. For any compact group G, the *integral* ${}^{\tau}K_G(G)$ is isomorphic to the *Verlinde algebra* of the theory of loop groups, at a certain level related to the twisting τ (see [F1] for the announcement and [F2] for a more detailed discussion). The proof will have wait for [FHT], but in §5, we give a canonical isomorphism between the complexifications of these objects, in terms of the Kac character formula. Finally, it is known that the Verlinde algebra encodes the dimensions of the spaces of "non-abelian θ-functions", and in §7 we incorporate this into the twisted K-theory framework.

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Part I. The twisted equivariant Chern character

1. The idea

Twisted K-theory ${}^{\tau}K_G^*(X)$ is a module over the untwisted $K_G^*(X)$, and in particular over the ring $K_G^0(*) = R_G$ of virtual complex representations of G. (Integer coefficients are understood, unless others are indicated). Similarly, ${}^{\tau}K_G^*(X;\mathbb{C}) = {}^{\tau}K_G^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ is a module over $R_G \otimes_{\mathbb{Z}} \mathbb{C}$, henceforth denoted $\mathbb{C}R_G$. The character identifies the latter with the ring of complex-valued algebraic class functions on G, or algebraic class functions over the complexification $G_{\mathbb{C}}$. This is (by definition) the ring of regular functions over the quotient variety $Q := G_{\mathbb{C}} /\!/ G_{\mathbb{C}}$ of *geometric invariant theory*. Its points correspond to the semi-simple conjugacy classes in $G_{\mathbb{C}}$, and also to the G-orbits of *normal* [5] elements in $G_{\mathbb{C}}$. When G is connected, Q is also the quotient $T_{\mathbb{C}}/W$ of the complex maximal torus by the Weyl group. Because Q is an affine variety, the ${}^{\tau}K_G^*(X;\mathbb{C})$ are spaces of global sections of sheaves of $\mathcal{O}$-modules ${}^{\tau}\mathcal{K}^*(X)$, obtained by Zariski localization over Q (see e.g. Ch. II of [H]). The sheaves are coherent, if ${}^{\tau}K_G^*(X;\mathbb{C})$ is a finite $\mathbb{C}R_G$-module; for instance, this is the case if X is a finite, G-equivariant CW-complex. If so, the main theorem (2.4) identifies, via the Chern character, the formal completions ${}^{\tau}\mathcal{K}^{0/1}(X)^{\wedge}_q$ of the stalks at a point $q \in Q$ with *twisted equivariant cohomologies* ${}^{\tau}H^{even/odd}_{Z(g)}(X^g;{}^{\tau}\mathcal{L}(g))$. Here, g is a normal group element associated to q, $Z(g) \subset G$ the unitary part of its centralizer and X^g the fixed-point set in X of (the unitary part of) the algebraic subgroup generated by g. The coefficients live in an equivariantly flat line bundle ${}^{\tau}\mathcal{L}(g)$ (2.11), which varies continuously with g. The only novelty is the twisting; in its absence, the ${}^{\tau}\mathcal{L}(g)$ are trivial and the result is well-known. (For torsion twistings on orbifolds, a closely related result was independently obtained in [LU]).

A detailed description of all these objects, for the group $G = \mathrm{SU}(2)$ acting on itself by conjugation, is discussed in Example (2.5).

Passage from ${}^{\tau}K_G^*(X;\mathbb{C})$ to sections of $\mathcal{K}^*$ is a global version of the *twisted Chern character* ${}^{\tau}ch: {}^{\tau}K_G^*(X;\mathbb{C}) \to {}^{\tau}H_G^*(X;\mathbb{C})$ (§3). Just as its untwisted version, this only sees the completion of K-theory at the augmentation ideal; this is a consequence of the Atiyah-Segal completion theorem [AS1]. The idea of "repairing" this problem by defining a global Chern character over G, while implicit in [S] (Prop. 4.1) and perhaps folklore, was proposed explicitly in [BBM] (and carried out for Abelian groups); [AS2] discussed finite groups, and a plethora of variations for compact groups followed ([BG], [DV], [G], [Ro]). We reprove the theorem here in its cleanest, algebraic form, as a spe-

[5] Those commuting with their hermitian adjoints

cial case of our twisted result.

Before proceeding, we should clarify why a reduction to twisted equivariant cohomology is helpful. Computation of the latter reduces to ordinary cohomology with coefficients in ${}^{\tau}\mathcal{L}(g)$ via an Atiyah-Hirzebruch spectral sequence (§3). Its analogue for twisted K-theory only yields the completion at the augmentation ideal: this is our completed stalk ${}^{\tau}\mathcal{K}^*(X)^{\wedge}_1$ at the identity[6], which is only part of the answer, and can vanish in interesting cases, such as ${}^{\tau}K_G(G)$ for semi-simple G. In this respect, equivariant ${}^{\tau}K$-theory for compact G can behave like the (untwisted) equivariant theory for a finite group, and the ${}^{\tau}\mathcal{K}^*$-sheaves can be skyscrapers supported at finitely many conjugacy classes.

One may ask whether our description of the completions determines ${}^{\tau}K^*_G$. In the sky-scraper case, the picture is completely satisfactory. In general, the functor taking a coherent sheaf to the product of its formally completed stalks is exact and fully faithful, but it takes some benevolence to declare ${}^{\tau}K^*_G$ known. A better approach to that problem proceeds via reduction to homogeneous spaces, leading to G-equivariant coherent sheaves $\mathcal{C}h^{0,1}(X)$ on $G_{\mathbb{C}}$, whose *invariant direct images* to Q are the $\mathcal{K}^{0,1}(X)$. This is prettier in two respects. First, $\mathcal{C}h$ is "less singular" than $\mathcal{K}$; for instance, when X is a point, the twisting defines a central extension of G by the circle group $\mathbb{T}$, and the sheaf $\mathcal{C}h^0$ is the associated line bundle; whereas $\mathcal{K}^0$ is a torsion-free sheaf of rank one, which can be singular where Q is so. Second, when suitably defined[7], the *derived fibers* $\mathcal{C}h^{0,1}(X)_g$ are naturally isomorphic to the K-theories of the fixed-point set X^g with ${}^{\tau}\mathcal{L}(g)$ coefficients, whereas the derived fibers of the $\mathcal{K}^*$ yield only the $Z(g)$-invariant parts of the same. (The case of finite groups provides a good illustration for this phenomenon: compare the formulations (2.1) and (2.2) below). Sadly, no immediate applications of this refined picture are known to us, which is why we refrain from including it here.

2. The main result

We assume that X is a finite CW-complex with G-action, in which case the definition of K-theory is uncontroversial. The main result (2.4) generalizes to other G-spaces, but may require a slight adjustment, depending on which version of K-theory is used; for instance, ${}^{\tau}K^*_G(X;\mathbb{C})\otimes_{\mathbb{C}R_G}(\mathbb{C}R_G)^{\wedge}_g$ must appear in lieu of the completion, if proper supports are used. Recall first the result for a finite group.

(2.1) Theorem ([AS2]). *For finite G,* $K^*_G(X;\mathbb{C})\cong\left[\bigoplus_{g\in G}K\left(X^g;\mathbb{C}\right)\right]^G$.

Here, G acts on the sum by conjugating the labels, and on X by translation. The right-hand side can be rewritten as a sum over the conjugacy classes q of G, with representatives $g(q)$:

$$K^*_G(X;\mathbb{C})\cong\bigoplus_q K\left(X^{g(q)};\mathbb{C}\right)^{Z(g(q))}. \tag{2.2}$$

The isomorphism (2.1) extends the complex-linear identification of representations with class functions by the character. Namely, for each $g\in G$, the restriction to X^g of a G-vector bundle V on X splits into eigen-bundles $V^{(\alpha)}$ under the fibrewise g-action, and (2.1) sends V to the complex linear combination $\sum\alpha\cdot V^{(\alpha)}$. More abstractly, let $\langle g\rangle\subset G$ be the subgroup generated by g; a class in $K^*_G(X)$ restricts to one in $K^*_{\langle g\rangle}(X^g)\cong R_{\langle g\rangle}\otimes K^*(X^g)$, and taking the trace of g on the first factor produces a $Z(g)$-invariant element in $K^*(X^g;\mathbb{C})$.

[6] This is an over-simplification, because the spectral sequence can be defined over the integers, where the augmentation completion sees more than our stalk at 1.

[7] In the derived category of coherent $\mathcal{O}$-modules.

Applying the Chern character gives two versions of the equivariant Chern isomorphisms:

$$\begin{aligned} ch{:}K_G^{0,1}(X;\mathbb{C}) &\xrightarrow{\sim} \Big[\bigoplus\nolimits_{g\in G} H^{even/odd}\big(X^g;\mathbb{C}\big)\Big]^G, \\ ch{:}K_G^{0,1}(X;\mathbb{C}) &\xrightarrow{\sim} \bigoplus\nolimits_q H^{even/odd}_{Z(g(q))}\big(X^{g(q)};\mathbb{C}\big). \end{aligned} \tag{2.3}$$

To describe their twisted analogues for arbitrary compact groups, we fix some notation. Since we take the algebraic route, we must discuss the complex group $G_{\mathbb{C}}$, even though all information will be contained in the unitary part G. Let $g \in G_{\mathbb{C}}$ be a normal element, generating an algebraic subgroup $\langle g\rangle_{\mathbb{C}} \subset G_{\mathbb{C}}$, with centralizer $Z_{\mathbb{C}}$. Normality of g ensures that these subgroups are the complexifications of their intersections $\langle g\rangle$ and $Z = Z(g)$ with G, that $\langle g\rangle$ is topologically cyclic, and that Z is its commutant in G. (This follows easily from the fact that normal elements are precisely those contained in the complexification of a *quasi-torus* in G, a subgroup meeting all components of G, whose neutral component is a maximal torus). As before, call $X^g \subset X$ the $\langle g\rangle$-fixed point set and ${}^{\tau}K_G^*(X;\mathbb{C})^{\wedge}_g$ the formal completion of twisted K-theory at the conjugacy class of g. Recall that the $\mathbb{C}R_G$-modules ${}^{\tau}K_G^*(X;\mathbb{C})$ are the spaces of sections of sheaves ${}^{\tau}\mathcal{K}^*(X)$ over $Q = G_{\mathbb{C}} /\!\!/ G_{\mathbb{C}}$; the completions of their stalks at g are the ${}^{\tau}K_G^*(X;\mathbb{C})^{\wedge}_g$. The following theorem generalizes the second isomorphism in (2.3), and the remainder of the section is devoted to its proof.

(2.4) Theorem. *The formal completions ${}^{\tau}K_G^{0,1}(X;\mathbb{C})^{\wedge}_g$ are isomorphic to the twisted cohomologies ${}^{\tau}H^{even,odd}_{Z(g)}\big(X^g;{}^{\tau}\mathcal{L}(g)\big)$, with coefficients in the flat line bundle ${}^{\tau}\mathcal{L}(g)$ of (2.11).*

(2.5) Example. Let $G = \mathrm{SU}(2)$, acting on $X = \mathrm{SU}(2)$ by conjugation. Then, $G_{\mathbb{C}} = \mathrm{SL}(2;\mathbb{C})$, and Q may be identified with the affine line, with coordinate q, as follows: the conjugacy class of the matrix $g = \mathrm{diag}(\lambda,\lambda^{-1})$ corresponds to the point $q = \lambda + \lambda^{-1}$, for $\lambda \in \mathbb{C}^{\times}$. These matrices form a complexified maximal torus $T_{\mathbb{C}}$, and the Weyl group S_2 interchanges λ and λ^{-1}. The closed interval $[-2,2]$ is the image of $\mathrm{SU}(2)$ in Q. The unitary centralizer $Z(g)$ is the maximal torus T, unless $q = \pm 2$, in which case it is the entire group G. The algebraic group $\langle g\rangle_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ generated by g is $T_{\mathbb{C}}$, and its unitary part $\langle g\rangle$ equals T, unless λ is a root of unity, in which case $\langle g\rangle = \langle g\rangle_{\mathbb{C}}$ is the finite cyclic group generated by g. The fixed-point set $X^g = T$, unless $\lambda = \pm 1$, in which case $X^g = G$. The twistings are classified by $H^3_G(G) \cong \mathbb{Z}$; we focus on the case $\tau \neq 0$. If $g \neq \pm I$, the flat line bundle ${}^{\tau}\mathcal{L}(g)$ over T has holonomy $\lambda^{2\tau}$ (see §4); so the cohomology in (2.4) is nil, unless $\lambda^{2\tau} = 1$. At $\lambda_k = \exp(k\pi i/\tau)$ ($k = 1,\dots,\tau-1$), the line bundle is trivial, but the differential which computes the twisted cohomologies in (2.4) is non-trivial, and is described as follows (cf. 3.4.iii). We can write $H^*_T(T;\mathbb{C}) \cong H^*(BT;\mathbb{C}) \otimes H^*(T;\mathbb{C}) \cong \mathbb{C}[[u,\theta]]/\theta^2$, where u is the coordinate on the Lie algebra of T (so $\lambda = \lambda_k \exp(u)$ is a local coordinate on T); the twisted cohomology group is then identified with the cohomology of the differential $2\pi\imath\theta$, and is one-dimensional. On the other hand, at $g = \pm I$, ${}^{\tau}\mathcal{L}(g)$ is trivial, but now $H^*_G(G;\mathbb{C})$ is identified with the subring of Weyl invariants in $H^*_T(T;\mathbb{C})$, which is $\mathbb{C}[[u^2,u\theta]]/(u\theta)^2$; the twisting differential $2\pi\imath\theta$ has zero cohomology, so there is no contribution from those points if $\tau \neq 0$. All in all, we obtain ${}^{\tau}K^0_G(G;\mathbb{C}) = 0$, ${}^{\tau}K^1_G(G;\mathbb{C}) = \mathbb{C}^{\tau-1}$, supported at the points $q_k = \lambda_k + \lambda_k^{-1}$. (When $\tau = 0$, a similar discussion shows that $\mathcal{K}^{0,1}$ are both locally free of rank one over Q; for a generalization of this result, see [BZ]).

Proof of (2.4). We start with a point $X = *$, when ${}^{\tau}K^0_G(*)$ is the module, also denoted ${}^{\tau}R_G$, of τ-

projective representations. More precisely, the class $\tau \in H^3_G(*;\mathbb{Z})$ defines an isomorphism class of central extensions of G by the circle group $\mathbb{T}$. Fixing such an extension $\tilde{G}$ — which can be viewed as a cocycle representative for τ — allows one to define the abelian group of those virtual representations of $\tilde{G}$ on which the central $\mathbb{T}$ acts naturally. This is the topologist's definition[8] of ${}^{\tau}K^0_G(*)$; it is evidently an R_G-module under tensor product. The complexification of $\tilde{G}$ defines an algebraic line bundle ${}^{\tau}\mathcal{C}h^0$ over $G_{\mathbb{C}}$, carrying a natural lifting of the conjugation action. Its fibers are the lines ${}^{\tau}\mathcal{L}(g)$ over $X = *$. Its invariant direct image to Q is a torsion-free sheaf ${}^{\tau}\mathcal{K}^0$; this need not be a line bundle, because the centralizers may act non-trivially on some fibres. Characters of τ-projective representations are invariant sections of ${}^{\tau}\mathcal{C}h^0$, and examining class functions on $\tilde{G}$ shows that the complexification ${}^{\tau}K^*_G(*;\mathbb{C})$ gets identified, in this way, with the space of invariant sections of ${}^{\tau}\mathcal{C}h^0$ (which is also the space of sections of ${}^{\tau}\mathcal{K}^0$). This completes the proof for a point.

(2.6) Proposition. *The restriction* ${}^{\tau}K^*_G(X;\mathbb{C})^{\wedge}_g \to {}^{\tau}K^*_Z(X^g;\mathbb{C})^{\wedge}_g$ *is an isomorphism.*

(2.7) Remark. This is closely related to Prop. 4.1 of [S], but is not quite equivalent to it. As the proof below shows, our proposition still holds if *étale localisation* (Henselisation) at g is used, instead of completion, but it usually fails for the usual (Zariski) localisation, even when X is a point: the fraction field of T is not equal to that of Q, which is the Weyl-invariant subfield.

Proof. Consider first the case when X is a point and $\tau = 0$. We are then asserting that the completed ring $(\mathbb{C}R_G)^{\wedge}_g$ of class functions is isomorphic, under restriction from G to Z, to $(\mathbb{C}R_Z)^{\wedge}_g$. This is true because $Z_{\mathbb{C}}$ is a local (étale) slice at g for the adjoint action of $G_{\mathbb{C}}$. For general τ, the sheaf ${}^{\tau}\mathcal{C}h^0$ on $Z_{\mathbb{C}}$ is the restriction of its G-counterpart, so the two direct images ${}^{\tau}\mathcal{K}^0$ on Q, coming from $G_{\mathbb{C}}$ and from $Z_{\mathbb{C}}$, agree near g.

To extend the result to a general X, it suffices, by a standard argument, to settle the case of a homogeneous space. In that case, ${}^{\tau}K^*_G(G/H) = {}^{\tau}K^*_H(*)$. The conjugacy class of g in $G_{\mathbb{C}}$ meets $H_{\mathbb{C}}$ in a finite number of classes, for which we can choose normal representatives $h_i = k_i^{-1} g k_i$, with $k_i \in G$, and get a natural isomorphism

$$(2.8) \qquad {}^{\tau}K^*_G(G/H;\mathbb{C})^{\wedge}_g \cong \bigoplus_i {}^{\tau}K^*_H(*;\mathbb{C})^{\wedge}_{h_i} \cong \bigoplus_i {}^{\tau}K^*_{k_i H k_i^{-1}}(*;\mathbb{C})^{\wedge}_g .$$

A coset $kH \in G/H$ is invariant under $\langle g \rangle$-translation iff $k^{-1}\langle g \rangle k \in H$. This holds precisely when $k^{-1}gk \in H_{\mathbb{C}}$; thus, $k^{-1}gk = hh_i h^{-1}$, for some $h \in H_{\mathbb{C}}$ and a unique i. As $k^{-1}gk$ and h_i are both normal, we can assume $h \in H$. As $khk_i^{-1} \in Z_{\mathbb{C}}$, and $k \in Z k_i H$, for a unique i. The fixed-point set of $\langle g \rangle$ on G/H is then the disjoint union over i of the subsets $Z k_i H$. These are isomorphic, as Z-varieties, to the homogeneous spaces $Z/Z \cap k_i H k_i^{-1}$. From here,

$$(2.9) \qquad {}^{\tau}K^*_Z\big((G/H)^g\big) \cong \bigoplus_i {}^{\tau}K^*_{Z \cap k_i H k_i^{-1}}(*),$$

and, as $Z \cap k_i H k_i^{-1}$ is the centralizer of g in $k_i H k_i^{-1}$, equality of right-hand sides in (2.8) and (2.9), after completion at g, follows from the known case when X is a point. ✠

We will now identify the completions in terms of cohomology. Let Y (soon to be X^g) be a Z-

[8] In the C^* algebra framework for K theory, this is a theorem.

space on which $\langle g\rangle$ acts trivially. Because $\langle g\rangle$ is topologically cyclic, $H^3_{\langle g\rangle}(*)=0$, so $\tau\in H^3_Z(Y)$ defines, in the Leray sequence $H^p_{Z/\langle g\rangle}(H^q_{\langle g\rangle}(Y))\Rightarrow H^{p+q}_Z(Y)$, a class

$$\mathrm{gr}(\tau)\in H^1_{Z/\langle g\rangle}\Big(Y;H^2_{\langle g\rangle}(*)\Big)=\mathrm{Hom}\Big(H_1^{Z/\langle g\rangle}(Y)\times\langle g\rangle;\mathbb{T}\Big). \tag{2.10}$$

(2.11) Definition. *The $Z/\langle g\rangle$-equivariant flat complex line bundle ${}^\tau\mathcal{L}(g)$ over Y is defined, up to isomorphism, by evaluating the complexification of the homomorphism (2.10) at g.*

(2.12) Construction. We can describe ${}^\tau\mathcal{L}(g)$ on the nose in the following two equivalent ways, after choosing a cocycle representing τ. The cocycle takes the form of a projective Hilbert space bundle $\mathbb{P}_Y$ over Y, with a projective-linear lifting of the Z-action. Over every point of Y, a central extension of the group $\langle g\rangle$ by the circle is defined from the action of $\langle g\rangle\subset Z$ on the fibers of $\mathbb{P}_Y$.

(i) The structural bundle of ${}^\tau\mathcal{L}(g)$ is the bundle of central $\mathbb{C}^\times$'s of the complexified extensions over $Y\times\{g\}$. Flatness is seen as follows. Since $\langle g\rangle$ is topologically cyclic, the extensions are trivial, but not *canonically* split. Two splittings differ by a homomorphism $\langle g\rangle_{\mathbb{C}}\to\mathbb{C}^\times$, and the latter form a discrete set. Following a splitting around a closed loop defines such a homomorphism, and its value at g gives the holonomy of ${}^\tau\mathcal{L}(g)$ around the loop.

(ii) The 1-dimensional characters of the centrally extended $\langle g\rangle$'s which restrict naturally to the centers form a principal bundle $p{:}H\to Y$ with fibre $H^2(B\langle g\rangle)$, equivariant under $Z/\langle g\rangle$. The isomorphism class of H is $\mathrm{gr}(\tau)$ in (2.10). Evaluation at g defines a homomorphism $H^2(B\langle g\rangle)\to\mathbb{C}^\times$; thereunder, the line bundle ${}^\tau\mathcal{L}(g)$ is associated to H.

(2.13) Remark. Note, in (ii), that the pull-back $\mathbb{P}_H:=p^*\mathbb{P}_Y$ to H has a *global* g-eigenspace decomposition: that is, we can define global sub-bundles labeled by the characters of $\langle g\rangle$, which split the Hilbert spaces associated to the fibres of $\mathbb{P}_H$. The twisted K-groups defined by any *non-empty* sub-bundle are naturally isomorphic to those defined by $\mathbb{P}_H$. The inclusion of the 1-eigenbundle $\mathbb{P}'_H\subset\mathbb{P}_H$, on which the $\langle g\rangle$-action is trivial, is a cocycle-level refinement of the fact that the class $p^*\tau\in H^3_Z(H)$ comes from a naturally defined τ' in $H^3_{Z/\langle g\rangle}(H)$. (We can ensure non-emptiness of the eigenbundle by arranging that the linear spaces associated to the fibres of $\mathbb{P}_Y$ contain all characters of $\langle g\rangle$).

Before completing the proof, we indicate a heuristic argument which illuminates the appearance of the flat line bundles. Translation by g on conjugacy classes defines an automorphism T_g of the algebra $\mathbb{C}R_Z$; this sends an irreducible representation V of Z, on which g must act as a scalar ξ, to $\xi\cdot V$. We can lift T_g to an intertwining automorphism of the module $K^*_Z(Y;\mathbb{C})$, by decomposing vector bundles and taking linear combinations in the same way. We would like to assert the following twisted analogue of this.

"Proposition". *T_g lifts to an intertwining $\mathbb{C}R_Z$-module isomorphism ${}^\tau K^*_Z(Y;\mathbb{C})\cong{}^\tau K^*_Z\big(Y;{}^\tau\mathcal{L}(g)\big)$.*

This would identify the completion of ${}^\tau K^*_Z(Y;\mathbb{C})$ at g with that of ${}^\tau K^*_Z\big(Y;{}^\tau\mathcal{L}(g)\big)$ at 1, leading, via the Atiyah-Segal completion theorem, to the equivariant cohomology in Thm. (2.4). The map from left to right should send a cocycle V to the complex linear combination $\sum\alpha\cdot V^{(\alpha)}$, defined from its

eigenspace decomposition under g. This time, however, $\langle g\rangle$ acts projectively, so the eigenvalues α are sections of a $\mathbb{C}^\times$-torsor, rather than complex functions on Y; so the output would be a "K-cocycle with values in ${}^\tau\mathcal{L}(g)$."

Unfortunately, no sensible general definition of *equivariant* K-theory with coefficients in flat line bundles seems to exist, and the "Proposition" is, at best, a definition. This is of no help in the proof of (2.4), so we must work around it. The following lemma gives the rigorous implementation of this eigenspace decomposition, with integer coefficients. Recall the twisting τ' defined in (2.13).

(2.14) Lemma. *The composition* ${}^{\tau'}K^*_{Z/\langle g\rangle}(H)_{cpt}\to{}^{\tau'}K^*_Z(H)_{cpt}\xrightarrow{p_*}{}^{\tau}K^*_Z(Y)$ *is an isomorphism (where the subscript indicates that the K-theory has compact vertical supports).*

Proof. The maps can be defined on K-cocycles. Functoriality of all the constructions allows us, by standard arguments, to reduce the question to the case when Y is a homogeneous Z-space, or equivalently (after replacing Z by the isotropy subgroup) to the case when Y is a point. The lemma then says ${}^\tau R_Z = {}^{\tau'}K^0_{Z/\langle g\rangle}(H)$; but, in view of our definition of τ', that simply expresses the decomposition of τ-projective Z-representations according to the projective character of the central element g. ✠

Remark. A similar argument shows that the inverse map is the g-invariant part of p^*.

We now complete the $\mathbb{C}R_Z$-modules in (2.14) at g, in two stages. Combining the completion theorem [AS1] with the twisted Chern character of §3 shows that tensoring with the completion of $\mathbb{C}R_Z$ along the subvariety $\langle g\rangle_{\mathbb{C}}\subset Z_{\mathbb{C}}//Z_{\mathbb{C}}$ converts the left-hand side into ${}^{\tau'}H^*_{Z/\langle g\rangle}(H;\mathbb{C})_{cpt}$. Now, completion at g can be performed fibrewise, with respect to the projection $p{:}H\to Y$. After identifying $(\mathbb{C}R_{\langle g\rangle})^\wedge_g$ with $H^*(B\langle g\rangle;\mathbb{C})$, this leads to ${}^{\tau'}H^*_{Z/\langle g\rangle}(Y;{}^\tau\mathcal{L}(g)\otimes H^*(B\langle g\rangle))$. But the last space is isomorphic to ${}^\tau H^*_Z(Y;{}^\tau\mathcal{L}(g))$; this is clear, for instance, in the Cartan model for equivariant cohomology. All in all, we have established the following natural isomorphism.

(2.15) Proposition. ${}^\tau K^*_Z(X^g;\mathbb{C})^\wedge_g\cong{}^\tau H^*_Z\left(X^g;{}^\tau\mathcal{L}(g)\right)$. ✠

Propositions (2.6) and (2.15) together imply theorem (2.4).

3. Twisted cohomology with complex coefficients

We now review the cohomological analogue of twisted K-theory. Let $(A^\bullet,d)$ be a commutative differential graded algebra model for the complex cocycles on the space X, with product denoted "$\wedge$". This can be de Rham's complex, if X is a manifold. Let $\tau\in H^3(X;\mathbb{C})$ be a complex twisting, and choose a cocycle $\eta\in A^3$ representing τ. We shall complete the cohomology ring with respect to its natural grading, so that, for instance, A^{even} is the *direct product* of the A^{2n}, and not the sum. This is only relevant when X is an *infinite CW*-complex (such as the classifying space BG of a compact group); if so, we must take care to define complex cohomology and K-theory of X as the limits over the finite subcomplexes of X; that is, we complexify the coefficients *before* computing cohomology.

(3.1) Definition. *The τ-twisted cohomologies* ${}^\tau H^*(X;\mathbb{C})$ *are the cohomology spaces of the complex* $\left(A^\bullet((\beta)),d+\beta\cdot\eta\wedge\right)$*, where the formal variable β has degree (-2).*

The complex in (3.1) is a differential graded module over $(A^\bullet,d)$, which makes ${}^\tau H^*(X;\mathbb{C})$ into a

graded module over the ordinary cohomology ring. For any $\omega \in A^2$, multiplication by $e^{-\beta\omega}$ identifies the cohomologies of $d + \beta\cdot\eta\wedge$ and $d + \beta\cdot(\eta + d\omega)\wedge$, which shows that, up to *non-canonical* automorphism, ${}^{\tau}H^*(X;\mathbb{C})$ depends on τ alone and not on η; the automorphisms come from the multiplicative action of exponentials of classes in $H^2(X;\mathbb{C})$.

(3.2) Proposition. *There exists a spectral sequence with* $E_2^{p,q} = H^p(X;\mathbb{C})$ *for even q, vanishing for odd q, with* $\delta_2 = 0$ *and* $\delta_3 = \tau\wedge$, *converging weakly to* ${}^{\tau}H^{p+q}(X;\mathbb{C})$. *It converges completely, if* $H^\bullet(X)$ *is finite-dimensional in each degree.*

We recall that "weak convergence" means that $E_\infty = \mathrm{gr}\, H^*$ for an induced filtration on H^*, under which the latter is complete; strong convergence means that H^* is Hausdorff; see [Mc], Ch. 3.

Proof. The filtration on $A^\bullet((\beta))$ by the powers of $\beta^{1/2}$ is complete and Hausdorff; this leads to a weakly convergent spectral sequence. Complete convergence follows, under our finiteness assumption, from the Mittag-Leffler conditions. ✠

(3.3) Remarks. (i) Additional twistings, by a flat line bundle on *X*, can be used; the same line bundle will appear in the cohomology coefficients, in (3.2).
(ii) When a compact group *G* acts on *X* and $\tau \in H^3_G(X;\mathbb{C})$, we define ${}^{\tau}H^*_G(X;\mathbb{C})$ to be the twisted cohomology of the Borel construction (homotopy quotient) $EG\times_G X$.

(3.4) Examples. (i) Assume that τ is a free generator of $H^*(X;\mathbb{C})$: that is, the latter is isomorphic to $R[\tau]$, for some graded ring *R*. Then, ${}^{\tau}H^*(X;\mathbb{C}) = 0$. Indeed, $E_4 = 0$ in (3.2).
(ii) For an example of (i), take *X* to be a compact connected Lie group *G*, and a τ which is non-trivial on $\pi_3(G)$. More relevant to us, with the same assumptions, is the homotopy quotient G/G for the adjoint action; the equivariant cohomology $H^*_G(G;\mathbb{C})$ breaks up as $H^*(BG;\mathbb{C})\otimes H^*(G;\mathbb{C})$.
(iii) Take $X = T$, a torus, $G = T$ acting trivially on *X*, and let $\tau\in H^2(BT;\mathbb{C})\otimes H^1(T;\mathbb{C})\subset H^3_T(T;\mathbb{C})$ be defined by a non-degenerate bilinear form on the Lie algebra $\mathfrak{t}$ of *T*. Then, ${}^{\tau}H^*_T(T;\mathbb{C}) = \mathbb{C}$, in degree $* = \dim T \pmod 2$. More precisely, ${}^{\tau}H^*(T;\mathbb{C})\cong H^*(T;\mathbb{C})$ (canonically, if τ factored as a cocycle), and the restriction ${}^{\tau}H^*_T(T;\mathbb{C})\to H^*(T;\mathbb{C})$ lands in top degree. This time, the E_3 term in (3.2) is a (sum of copies of the) Koszul complex $\Lambda\mathfrak{t}^*\otimes S\mathfrak{t}^*$, with differential $\tau\wedge$.

(3.5) Proposition. *There exists a natural* twisted Chern character ${}^{\tau}ch\colon {}^{\tau}K^*(X;\mathbb{C})\to{}^{\tau}H^*(X;\mathbb{C})$, *which is a module isomorphism over the Chern isomorphism* $ch\colon K^*(X;\mathbb{C})\to H^*(X;\mathbb{C})$.

Proof. In the classifying space interpretations $B\mathrm{U}(1)$ and $B\mathrm{U}(\infty)$ of the projective unitary group $\mathbb{P}\mathrm{U}$ and the space $\mathbb{F}^0$ of Fredholm operators of index 0, the Ad-action classifies the tensor product of the universal line bundle Det with the universal vector bundle. The Chern character identifies the rational homotopy type $\mathbb{F}^0\otimes\mathbb{Q}$ with the product of Eilenberg-MacLane spaces $K(\mathbb{Q};2n)$ $(n>0)$, and the Ad-action becomes $\omega\otimes\varphi\mapsto\exp(\omega)\wedge\varphi$, where ω is a 2-cocycle and φ an even one. Recall that the (negative) ${}^{\tau}K^*$ are the homotopy groups of the space $\Gamma(X;\mathbb{F}^0_X)$ of sections of a Fredholm Ad-bundle $\mathbb{F}^0_X$ classified by τ; morally, the term $\beta\eta\wedge$ in the twisted differential (3.1) is the connection form on the bundle of coefficients $\mathbb{F}^0_X\otimes\mathbb{Q}$ over *X*. This is made precise in the differential graded Lie algebra (DGLA) model for the rational homotopy type $\Gamma(X;\mathbb{F}^0_X)\otimes\mathbb{Q}$.

Because the action of $\mathbb{P}\mathrm{U}$ on $\mathbb{F}^0$ fixes the identity, its rational homotopy type is captured by

a split extension, in the category DGLAs,

$$0 \to \pi_\bullet \mathbb{F}^0 \otimes \mathbb{Q} \to E_\bullet \to \pi_3(B\mathbb{P}\mathrm{U}) \otimes \mathbb{Q} \to 0, \tag{3.6}$$

which represents the fibre bundle $\mathbb{F}_X^0 \otimes \mathbb{Q}$ in the universal case when $X = K(\mathbb{Q};3)$, $\tau = 1 \in H^3(X;\mathbb{Q})$. An obvious homotopical invariant in (3.6) is the bracket $b: \pi_2(\mathbb{P}\mathrm{U}) \otimes \pi_{2m}(\mathbb{F}^0) \to \pi_{2m+2}(\mathbb{F}^0)$. It is induced by a homotopical realization of the map "Ad − Id": $\mathbb{P}\mathrm{U} \times \mathbb{F}^0 \to \mathbb{F}^0$. On classifying bundles, the relevant map is tensoring with $(\mathrm{Det} - 1)$, which restricts to the Thom class on any generating sphere in $\mathbb{P}\mathrm{U}$. Thus, restricted to the generator of $\pi_2(\mathbb{P}\mathrm{U})$, b is the Bott periodicity isomorphism β.

In view of this bracket, the obvious candidate for $E_\bullet$ (but not the only one) is the semi-direct sum $\pi_3(B\mathbb{P}\mathrm{U}) \oplus \pi_\bullet(\mathbb{F}^0)$ with bracket b. Pulling back to arbitrary X would imply (using Sullivan's determination of the rational homotopy of a space of sections) that $\Gamma(X;\mathbb{F}_X^0) \otimes \mathbb{Q}$ was represented by the positive homology degree part of the abelian DGLA $A_\bullet(X) \otimes \pi_\bullet \mathbb{F}^0$, with differential $d + \beta \cdot \eta \wedge$ (where we have set $A_k := A^{-k}$). As the Chern character converts the Bott isomorphism into the self-identifications of the $K(\mathbb{Q};2n)$, it follows that the homotopy groups of the resulting candidate for $\Gamma(X;\mathbb{F}_X^0) \otimes \mathbb{Q}$ are isomorphic to the negative cohomology degree groups of (3.1), under ch.

Thus, proposition (3.5) reduces to the statement that the obvious candidate for (3.6) is the correct one. In fact, the semi-direct sum describes the *only* action of $\mathbb{P}\mathrm{U} \otimes \mathbb{Q}$ on $\mathbb{F}^0 \otimes \mathbb{Q}$ compatible with b, which preserves the addition (loop space) structure on $\mathbb{F}^0 \otimes \mathbb{Q}$, as our action must do. Indeed, such actions are classified by maps from $K(\mathbb{Q};3)$ to $B\mathrm{Aut}(\mathbb{F}^0 \otimes \mathbb{Q})$ up to homotopy; or again, by π_3 of the latter, which is also $\pi_2 \mathrm{Map}(B\mathbb{F}^0 \otimes \mathbb{Q}, B\mathbb{F}^0 \otimes \mathbb{Q})$. But an element in last group is defined precisely by a collection of maps $\pi_{2m}(\mathbb{F}) \to \pi_{2m+2}(\mathbb{F})$, and these are the ones detected by bracketing with the generator of $\pi_2(\mathbb{P}\mathrm{U})$. ✠

(3.7) Remark. Another, more elegant proof of (3.5) follows from the existence, outlined in [F2], of a cocycle-level twisted Chern character from ${}^\tau K$-cocycles to differential forms, which specializes to the usual one, when the twisting can be trivialized. There results a functorial homomorphism between the theories, which is an isomorphism over small enough sets (where τ is trivial); the Mayer-Vietoris principle implies that it is a global isomorphism.

Part II. A special case: computation of ${}^\tau K_G^*(G;\mathbb{C})$

4. Reduction to the maximal torus

Let $\tau \in H_G^3(G)$ be an integral class which restricts trivially to $H^3(T)$, the maximal torus. Call τ *non-singular* when its restriction to $H_T^3(T)$, viewed as a linear map $H_2(BT) \to H^1(T)$, has full rank; we shall assume this to be so. The last map defines, after tensoring with $\mathbb{T}$, an isogeny $\lambda: T \to T^\vee$ to the dual torus $T^\vee$. The kernel of λ is a finite subgroup $F \subset T$. Interpreting points in $T^\vee$ as isomorphism classes of flat line bundles on T, we observe the following.

(4.1) Proposition. *The flat line bundle ${}^\tau\mathcal{L}(t)$ over T, associated to τ in (2.11), is $\lambda(t)$.* ✠

Using Theorem (2.4), this suffices to determine ${}^\tau K_G^*(G;\mathbb{C})$ in some important special cases.

(4.2) Theorem. *(i) For the trivial action of T on itself, ${}^\tau\mathcal{K}^*(T)$ is a sky-scraper sheaf with one-dimensional stalks supported at the points of F, in dimension* $\dim T$ *mod 2.*

(ii) Let G be connected, with $\pi_1(G)$ torsion-free. For the conjugation action of G on itself, ${}^\tau\mathcal{K}^*(G)$ is a skyscraper sheaf on $Q = T_{\mathbb{C}}/W$ with 1-dimensional stalks, in dimension $d = \dim G \bmod 2$, supported at the regular *Weyl orbits in F.*

Proof. This follows from Examples (3.4.ii) and (iii), together with part (i) of the following. ✠

(4.3) Lemma. *Let G be connected.*
(i) If $\pi_1(G)$ is torsion-free, the centralizer of any element is connected.
(ii) In general, for any $g \in G$, $\pi_0 Z(g)$ is naturally a subgroup of $\pi_1(G)$.
(iii) If g is regular, $\pi_0 Z(g)$ is a subgroup of the Weyl group of G, natural up to conjugation.

Proof. (i) The short exact sequence $0 \to \pi_1(G') \to \pi_1(G) \to \pi_1(G^{ab}) \to 0$, where $G' := [G,G]$ is the commutator subgroup and G^{ab} the abelianization of G, shows that the torsion subgroup of $\pi_1(G)$ is $\pi_1(G')$; hence, G' is simply connected. The neutral component Z^0 of the center of G surjects onto the quotient G^{ab}, because the two Lie algebras are isomorphic and G^{ab} is connected; so $G = G'Z^0$. Translating by Z^0 shows that it suffices to check the assertion for G'; but this is a result of [Bo].
(ii) Write $G = \tilde{G}/\pi$, where the central subgroup π in the covering group $\tilde{G}$ is isomorphic to the torsion subgroup of $\pi_1(G)$, and $\pi_1(\tilde{G})$ is torsion-free. The $Z(g)$-conjugate of a lifting $\tilde{g}$ of g is another lift of g, and this defines a homomorphism from $Z(g)/Z(g)^0$ to π. If $z\tilde{g}z^{-1} = \tilde{g}$, then z lifts to an element of the centralizer of $\tilde{g}$ in $\tilde{G}$; but the latter is connected, by (i), so z must lie in $Z(g)^0$, and our homomorphism is injective.
(iii) Clearly, $Z(g)$ must normalize the unique maximal torus containing g, so $Z(g)/Z(g)^0$ embeds into the normalizer of that torus, which is the Weyl group, up to conjugation. ✠

(4.4) Remark. There always exists a regular conjugacy class whose fundamental group (which is π_0 of the centralizer) is the full torsion subgroup of $\pi_1(G)$. When G is simple, such a point is $\exp(\rho/c)$, where ρ is the half-sum of the positive roots, c the dual Coxeter number of $\mathfrak{g}$, and $\mathfrak{t}$ is identified with $\mathfrak{t}^*$ via the *basic* inner product, which matches the long roots with the short coroots. In general, the product of the corresponding points in the simple factors of G has this property.

As in the case of torsion-free π_1, singular conjugacy classes do not contribute to ${}^\tau K_G(G;\mathbb{C})$ (the relevant twisted cohomology vanishes when the centralizer contains an SU_2). The contribution at a regular point $f \in T$ is the space of invariants under $Z := Z(f)$ in ${}^\tau H_T^*(Z; {}^\tau\mathcal{L}(f))$. Each component Z^w of Z (labeled by a Weyl group element, as in 4.3.iii) is now a torus, and contributes to cohomology only if ${}^\tau\mathcal{L}(f)$ has trivial holonomy. If τ is non-singular, this will happen at isolated points f. If so, the generator of ${}^\tau H_T^*(Z^w; {}^\tau\mathcal{L}(f))$ is then the fibre of ${}^\tau\mathcal{L}(f)$ tensored with the volume form on Z^w/T (which is also the one on the invariant subspace $\mathfrak{t}^w$). All in all, we get one line from Z^w when ${}^\tau\mathcal{L}(f)$ is isomorphic to $\det(\mathfrak{t}^w)$, as a Z-equivariant line bundle over Z^w, and zero otherwise. So ${}^\tau\mathcal{K}^*(G)$ is still a sum of sky-scraper sheaves, but it is difficult to give a clean, explicit general expression for the dimensions of its stalks. Instead, a direct relation to the representations of the loop group of G can be given in terms of the Kac character formula, whose numerator is a distribution supported at distinguished conjugacy classes in a subgroup of LG isomorphic to Z. However, as the necessary general results do not seem to appear in the literature in the precise form we need, we shall confine the discussion in the next section, to the case when π_1 has no torsion (when $Z = T$).

5. The Kac numerator and the ${}^{\tau}K$-class of an LG-representation

Our description of ${}^{\tau}K_G^*(G;\mathbb{C})$ leads to a concrete, if intriguing, isomorphism with the complexified space of *positive energy representations* (PERs) of the loop group LG at a certain shifted level, which relates ${}^{\tau}K$-classes to distributional characters. The discussion that follows is quite crude, as it ignores the *energy* action on representations, so that we only see the value of the Kac numerator at $q=1$; detecting the q-powers requires the $\mathbb{T}$-equivariant version of ${}^{\tau}K_G^*(G)$, which will be treated elsewhere.

The representations that concern us are projective, and the relevant central extensions of LG by $\mathbb{T}$ turn out to be classified by their topological *level* in $H_G^3(G)$, arising form the connecting homomorphism ∂ in the exponential sequence for group cohomology with smooth coefficients,

$$H^2_{LG}(\mathbb{R})\to H^2_{LG}(\mathbb{T})\xrightarrow{\partial} H^3_{LG}(\mathbb{Z})\to H^3_{LG}(\mathbb{R}), \tag{5.1}$$

and the identification, for connected G, of $H^3_{LG}(\mathbb{Z})$ with $H^3_G(G)$. (For connected G, $BLG=LBG$, and the latter is the homotopy quotient G/G under the adjoint action). When G is semi-simple, the outer terms vanish ([PS], Ch. 14) and the topological level completely determines the extension. For any G, the levels that are relevant to us restrict trivially to $H^3(T)$; their restriction to $H^3_T(T)$ define Weyl-invariant, integral bilinear forms on the integer lattice of T. From a bilinear form B, an extension of the Lie algebra $L\mathfrak{g}$ is defined by the 2-cocycle $(\xi,\eta)\mapsto \mathrm{Res}_{z=0}B(\xi,d\eta/dz)$, and the remaining information about the group extension is contained in the torsion part of the level.

A distinguished topological level, the *adjoint shift* σ, is the pull-back under the adjoint representation $\mathrm{Ad}:G\to SO(\mathfrak{g})$ of the element $(1,1)$ in $H^3_{SO}(SO)=\mathbb{Z}\oplus\mathbb{Z}/2$. (The splitting comes from the inclusion of the identity in SO; the free summand has a distinguished positive direction, for which the associated bilinear form B is positive definite; the non-trivial torsion element is the integral lift $W_3\in H^3_{SO}(e)$ of the third Stiefel-Whitney class.) When G is simple and simply connected, $H^3_G(G)\cong H^4(BG)\cong\mathbb{Z}$ and $\sigma=c$, the dual Coxeter number.

There is also a component $\sigma'\in H^1_G(G;\mathbb{Z}/2)$ of the adjoint shift, pulled back by Ad from the non-trivial element of $H^1_{SO}(SO;\mathbb{Z}/2)=\mathbb{Z}/2$. On the loop group side, this is a homomorphism from LG to $\mathbb{Z}/2$ (or *grading*, cf. Appendix A). The presence of this grading means that the "usual" Verlinde algebra is really a ${}^{\sigma',\tau}K$-theory, as in Appendix A; the ${}^{\tau}K$-theory, on the other hand, corresponds to a graded version of the Verlinde algebra, built from the σ'*–graded* representations of the loop group (see the definitions A.1 of the modules of graded representations of a graded group).

(5.2) Remark. The adjoint shift is best understood in terms of the positive energy spinors on $L\mathfrak{g}$. The central extension σ of the loop group arises by pulling back the Spin representation of $LSO(\mathfrak{g})$; similarly, the grading is pulled back from the Clifford algebra on $L\mathfrak{g}$. Without unduly labouring this point here, we mention that the truly canonical loop group counterpart of ${}^{\tau}K_G^*(G)$ is not the Verlinde algebra, but the K-group of graded, τ-projective PERs of the crossed product $LG\ltimes\mathrm{Cliff}(L\mathfrak{g})$. The adjoint and dimension shifts arise when relating the latter to PERs of the loop group, by factoring out the Spin representation.

(5.3) Remark. When $\pi_1(G)$ has no torsion, the only truly conspicuous part of the adjoint shift is the dual Coxeter number. Indeed, $\mathrm{Ad}^*(W_3)=0$; further, if τ is non-singular (which we always assume),

we do not err too much by ignoring the H^1-component σ', in the following sense. It turns out that all PER's of the loop group can be σ'-graded, so the usual and the σ'-graded Verlinde algebras are isomorphic (though some sign choices are involved in an isomorphism). See also Remark (5.5). The last statement usually fails when π_1 has torsion, e.g. for $G = \mathrm{SO}(3)$.

By definition, *positive energy* representations of LG carry an intertwining *energy* action of the circle group of rotations of the loop, with spectrum bounded below. Denoting the energy variable by q, irreducible PERs admit *formal characters* in $R_G((q))$. We focus on the case when $\pi_1(G)$ has no torsion; on the loop group side, this ensures that representations are determined by their (q–)restriction to T. The corresponding topological fact is the injectivity of the restriction morphism[9] ${}^{\tau}K_G^d(G) \to {}^{\tau}K_T^d(T)$. The restriction to T of the formal character of the irreducible PER of level χ and zero-energy space of highest weight $\lambda \in \mathfrak{t}^*$ is given by the *Kac formula*:

$$\frac{\sum_{\mu \in W_{aff} \cdot (\lambda+\rho)} (-1)^{\ell(\mu)} \exp(\mu) \cdot q^{\beta(\mu,\mu) - \beta(\lambda+\rho,\lambda+\rho)}}{\prod_{\alpha>0} \big(\exp(\alpha/2) - \exp(-\alpha/2)\big) \cdot \prod_{n>0;\alpha} \big(1 - q^n \exp(\alpha)\big)}, \tag{5.4}$$

where β is the bilinear form on $\mathfrak{t}^*$ associated to the level $\tau = \chi + \sigma$, α ranges over the roots of $\mathfrak{g}$ and μ over the affine Weyl orbit of $\lambda + \rho$; $\ell(\mu)$ is the length of the affine Weyl element w taking μ to $\lambda + \rho$. The *Kac denominator* in (5.4) is the formal super-character of the spinors on $L\mathfrak{g}/\mathfrak{t}$.

(5.5) Remark. Two subtleties are concealed in (5.4). First, the character is not a function, but a section of the line bundle $\mathcal{O}(\chi)$ over LG, associated to the central extension at level χ. Similarly, the numerator is the section of $\mathcal{O}(\tau)$. These central extensions are of course trivial over T, but not canonically so; and the weights λ, ρ and μ of T are *projective weights*, of levels χ, σ and τ, respectively. Second, in the presence of an H^1 component σ' of the adjoint shift, the formula (5.4) gives the *super-character* of a graded PER; the usual character is obtained by modifying the signs in the numerator of (5.4) by the sign representation of the affine Weyl group defined by σ'.

The following is obvious by inspection.

(5.6) Proposition. *The numerator in (5.4), at $q = 1$, is a distribution on T; more precisely, a Weyl anti-invariant linear combination of δ-sections of $\mathcal{O}(\tau)$, based at the regular points of F.* ✠

There results an obvious isomorphism between ${}^{\tau}K_G^d(G;\mathbb{C})$ and the complex span of irreducible PERs at level χ: the "value" of our K-class at any $f \in F^{\mathrm{reg}}$ gives the coefficient of the δ-function based there. To read off this value, we identify ${}^{\tau}K_G^d(G;\mathbb{C})^{\wedge}_f$ with ${}^{\tau}H_T^*(T; {}^{\tau}\mathcal{L}(f))$, as in §4, restrict to ${}^{\tau}H^{top}(T; {}^{\tau}\mathcal{L}(f))$ and integrate over T; the answer takes values in the fibre of $\mathcal{O}(\tau)$ over f. The last integration step also accounts for the Weyl *anti*-invariance of the answer.

6. The Pontryagin product on ${}^{\tau}K_G^*(G)$ by localization

We now define and compute the Pontryagin product on ${}^{\tau}K_G^*(G;\mathbb{C})$, under the simplifying assumption that G is connected and $\pi_1(G)$ is torsion-free. There is a good *a priori* reason (6.3) why the an-

[9] We proved it in §4 for the complexifications, but the statement holds integrally, see [FHT]

swer is a quotient of $\mathbb{C}R_G$, but we shall make the homomorphism explicit by localization, and recover a well-known description (6.4) of the complex Verlinde algebra.

In the notation of §4, comparing the sequence $1\to Z^0\to G\to \mathrm{Ad}(G')\to 1$ with the sequence of §4 leads to a *rational* splitting of $H^3_G(G)$,

$$H^3_G(G)\otimes\mathbb{Q}=\left[H^3_{G'}(G')\oplus H^3(G^{ab})\oplus H^2(BG^{ab})\otimes H^1(G^{ab})\right]\otimes\mathbb{Q}. \tag{6.1}$$

If our twisting τ vanishes in $H^3(T)$, then its middle component is null. In this case, τ is equivariantly *primitive* under the multiplication map $m{:}G\times G\to G$, meaning that $m^*\tau=(\tau,\tau)$, the latter denoting the restriction to $H^3_G(G\times G)$ of $\tau\boxtimes 1+1\boxtimes\tau$ in $H^3_{G\times G}(G\times G)$. A priori, this holds only rationally, but the absence of torsion in $H^3_G(G)$ shows that equality holds over $\mathbb{Z}$. Using the Künneth and restriction maps ${}^\tau K^*_G(G)\otimes{}^\tau K^*_G(G)\to{}^{(\tau,\tau)}K^*_{G\times G}(G\times G)\to{}^{(\tau,\tau)}K^*_G(G\times G)$, we define the *Pontryagin (convolution) product* on ${}^\tau K^*_G(G)$ as the push-forward along m:

$$m_!\colon {}^\tau K^*_G(G)\otimes{}^\tau K^*_G(G)\to{}^\tau K^*_G(G). \tag{6.2}$$

The absence of torsion in H^3 implies the vanishing of the Stiefel-Whitney obstruction W_3 to Spin^c-orientability, ensuring that $m_!$ can be defined. However, choices are involved when $\pi_1(G)\neq 0$, and the map (6.2) is only defined up to tensoring with a 1-dimensional character of G. There are two sources for this ambiguity, and they are resolved in different ways.

The first, and more obvious ambiguity lies in a choice of Spin^c structure. This is settled by choosing an Ad-invariant Spin^c structure on $\mathfrak{g}$; but the true explanation of this ambiguity, and the correct way to remove it, emerges from the use of the twisted K-homology ${}^\tau K^G_0(G)$, in which push-forward is the natural map. The Spin^c lifting ambiguity gets transferred into the Poincaré duality identification of ${}^\tau K^G_0(G)$ with ${}^\tau K^d_G(G)$ ($d=\dim G$).

The second, and more subtle indeterminacy, comes from the need to represent τ by a cocycle, in order to fix the ${}^\tau K$-group; otherwise, ${}^\tau K^*_G(G)$ is only known up to multiplication by a K-theory unit. To remove this ambiguity in (6.2), we need to lift the equality $m^*\tau=(\tau,\tau)$ from cohomology classes to 3-cocycles. When τ is transgressed from an integral four-dimensional class on BG, there is a distinguished such lifting, and a canonical multiplication results [FHT]. A natural multiplication also exists, for any class τ, when π_1 has no torsion, for a different reason. As $H^3_G(\{e\})=0$, we can represent τ by a cocycle vanishing on $BG\times\{e\}$. This defines an isomorphism $R_G\cong{}^\tau R_G$ and, after a choice of Spin^c structure, a direct image $e_!\colon R_G\to{}^\tau K^d_G(G)$. We normalize (6.2) by declaring $m_!(e_!1\otimes e_!1)=e_!1$. Clearly, this makes ${}^\tau K^d_G(G)$ into an R_G-algebra, and our normalization is the only one with this property.

(6.3) Remark. When π_1 has no torsion, $e_!$ is surjective, so ${}^\tau K^d_G(G)$ is a quotient of R_G, and the multiplication is determined by this. But we still want to "see" it by localization.

(6.4) Theorem. *${}^\tau K^*_G(G;\mathbb{C})$, with the Pontryagin product (6.2), is isomorphic to the ring of Weyl-invariant functions on the* regular *points of the subset F of (4.2), with pointwise multiplication.*

(6.5) Remark. There seems to be no description of comparable elegance for the multiplication when $\pi_1(G)$ has torsion.

Proof. The map (6.2) turns ${}^{\tau}K_G^d(G)$ into an R_G-algebra. Localizing over $\mathrm{Spec}(\mathbb{C}R_G)$, as in (4.2), we obtain 1-dimensional fibres over the regular elements of $F/W \subset T/W$. This already proves the theorem; but we can exhibit the "localized convolution product" concretely, in relation to the isomorphism in §5. At a regular point $f \in F$, whose centralizer, by (4.3), is T, the localized K-theory is isomorphic to ${}^{\tau}H_T^*(T;\mathbb{C})$. Dividing by the K-theoretic Euler class for the inclusion $T \subset G$ — which is the Weyl denominator — converts G-convolution to T-convolution. As noted in (3.4.iii), the generator of ${}^{\tau}H_T^*(T;\mathbb{C})$ sees the volume form on T, upon forgetting the T-action; so this last convolution algebra is isomorphic to $\mathbb{C}$. The isomorphism in (6.4) is related to the one in §5 by division by the Weyl denominator. ✠

7. Verlinde's formula as a topological index in ${}^{\tau}K$-theory

The *Verlinde formula* expresses the dimension of the spaces of sections of positive holomorphic line bundles over the moduli space M of semi-stable $G_{\mathbb{C}}$-bundles over a compact Riemann surface Σ of genus $g > 0$. A version of the formula exists for all semi-simple groups [AMW], but we confine ourselves to simple, simply connected ones, in which case $\mathrm{Pic}(M) = \mathbb{Z}$, line bundles being classified by their Chern class in $H^2(M)$. This embeds in $H^3_G(G) \cong \mathbb{Z}$, and it is in the latter[10] that we shall measure it. Positive line bundles have no higher cohomology [KN], and the dimension of their space of sections is

$$h^0\big(M;\mathcal{O}(h)\big) = \chi\big(M;\mathcal{O}(h)\big) = |F|^{g-1} \sum_{f \in F^{\mathrm{reg}}/W} \Delta(f)^{2-2g}. \tag{7.1}$$

where the group F of §4 is defined with respect to the shifted level $h + c$, and Δ is the anti-symmetric (spinorial) Weyl denominator. We shall replicate the right-hand side of (7.1) in twisted K-theory. This does *not* give yet another proof of the Verlinde formula; rather, it interprets it as an infinite-dimensional index theorem, in which ${}^{\tau}K$-theory carries the *topological index*.

Reinterpretation of the left side of (7.1)

Let $\Sigma^{\times}$ be the complement of a point $z = 0$, in a local coordinate z on Σ, $G((z))$ the "formal loop group" of Laurent series with values in $G_{\mathbb{C}}$ and $G[\Sigma^{\times}]$ the subgroup of $G_{\mathbb{C}}$-valued algebraic maps on $\Sigma^{\times}$, and let $X := G((z))/G[\Sigma^{\times}]$ be a *generalized flag variety* for $G((z))$. X is related to M via the quotient stack $X/G[[z]]$ by the group of formal regular loops, which is also the moduli stack of algebraic G-bundles on Σ. (For more background on these objects, we refer to [BL1], [Fa], [LS]).

(7.2) Remark. If Σ° is the complement $\Sigma - \Delta$ of a small open disk centered at $z = 0$, $LG_{\mathbb{C}}$ the smooth loop group based on $\partial\Delta$ and $Hol(\Sigma^{\circ}, G_{\mathbb{C}})$ the subgroup of loops extending holomorphically over Σ°, X is an algebraic model for the homogeneous space $X' := LG_{\mathbb{C}}/Hol(\Sigma^{\circ}, G_{\mathbb{C}})$, which is dense in X and homotopy equivalent to it [T1].

Algebraic line bundles over X are classified by their Chern class in $H^2(X) \cong H^3(G) \cong \mathbb{Z}$ [T1]. Theorem 4 of *loc.cit.* asserts that $H^0(X;\mathcal{O}(h))$ is a finite sum of duals of irreducible PERs of $G((z))$ of level h, whereas higher cohomologies vanish. The multiplicity of the vacuum representation is given by (7.1); more generally, the dual $\mathcal{H}(V)^*$ of the PER with ground space V appears m_V

[10] The two groups are not always equal, cf. [BL2]; but all elements of the second do determine *reflexive sheaves* on M, and Verlinde's formula then expresses their holomorphic Euler characteristic

times, where

$$m_V = |F|^{g-1} \sum_{f \in F^{\mathrm{reg}}/W} \Delta(f)^{2-2g} \cdot \chi_V(f). \tag{7.3}$$

We can reformulate this, using the inner product[11] on the Verlinde algebra $V(h)$, in which the irreducible PERs form an orthonormal basis. On functions on F^{reg}/W, this product is given by

$$\langle \varphi | \psi \rangle = |F|^{-1} \sum_{f \in F^{\mathrm{reg}}/W} \Delta(f)^2 \cdot \overline{\varphi}(f) \psi(f). \tag{7.4}$$

Formulae (7.1), (7.3) are then succinctly captured by the following identity in the Verlinde algebra:

$$\chi\big(X; \mathcal{O}(h)\big)^* = H^0\big(X; \mathcal{O}(h)\big)^* = |F|^g \cdot \Delta^{-2g}. \tag{7.5}$$

Twisted K meaning of the right-hand side of (7.5)

Consider the *product of commutators* map $\Pi: G^{\times 2g} \to G$ defined by Σ°: viewing G^{2g} and G as the moduli spaces of based, flat G-bundles over Σ°, resp. $\partial\Sigma^\circ$, Π is the restriction of bundles to the boundary. It is equivariant for G-conjugation; dividing by that amounts to forgetting the base-point, but we shall not do so, and work equivariantly instead. (Reference to a base-point can be removed by using the *moduli stacks* of unbased flat bundles, the quotient stacks G^{2g}/G and G/G).

The twisting $\tau = h + c \in H^3_G(G) \cong \mathbb{Z}$ lifts trivially to $H^3_G(G^{2g})$, and is canonically trivialized as follows. Any τ is transgressed from a class $\tau' \in H^4(BG)$, under the tautological classifying map $\partial\Sigma^\circ \times G/G \to BG$; in any reasonable model for cocycles, $\Pi^*\tau$ is the co-boundary of the slant product with Σ° of the pull-back of τ' under $\Sigma^\circ \times G^{2g}/G \to BG$. So we have a natural isomorphism $K^*_G(G^{2g}) = {}^\tau K^*_G(G^{2g})$; in particular, a canonical class "${}^\tau 1$" is defined in ${}^\tau K^0_G(G^{2g})$.

(7.6) Theorem. *$\Pi_!({}^\tau 1) \in {}^\tau K^d_G(G)$ is the right-hand side of (7.5). In other words, the multiplicities of $\Pi_!({}^\tau 1)$ in the basis of PERs are the Verlinde numbers (7.3).*

Proof. Localizing to a regular $f \in F/W$, $\Pi_!({}^\tau 1)$ agrees with the push-forward of ${}^\tau 1/\Delta(f)^{2g}$ along T^{2g}, where $\Delta(f)^{2g}$ is the relative K-theory Euler class of $T^{2g} \subset G^{2g}$. T^{2g} maps to $e \in G$, so $\Pi_!$ factors as the push-forward to $K^0_T(*)$, followed by $e_!$ of §6. It may seem at first glance that the K-theoretic integral of ${}^\tau 1$ over T^{2g} is nought, but that is not so. When factoring $\Pi_!$, we have trivialized $\Pi^*\tau$ on T^{2g}, by doing so first in $H^3_G(e)$. This differs from the Σ°-transgression trivialization, coming from G^{2g}: the difference is the 2-dimensional transgression of $\tau' \in H^4(BG)$ over Σ, via the tautological classifying map $\Sigma \times T^{2g} \to BT$. The associated line bundle over T^{2g} is the restriction of $\mathcal{O}(h+c)$, and its integral over T^{2g} is $|F|^g$ [AMW]. All in all, the T-restriction of our push-forward is $|F|^g \cdot \Delta(f)^{-2g} \cdot e_!(1)$; but this is the right side of (7.5), evaluated at f. ✠

Interpretation as an index theorem

We have the following set-up in mind. Let $\pi: S \to B$ be a proper submersion of manifolds of relative dimension d, $\tau \in H^3(B)$ an integral class whose lift to S is null. Representing τ by a cocycle defines the twisted groups ${}^\tau K^*(B)$, ${}^{\pi^*\tau} K^*(S)$, and expressing $\pi^*\tau$ as a co-boundary $\delta\omega$ fixes an isomorphism ${}^{\pi^*\tau} K^*(S) \cong K^*(S)$. As a result, a class ${}^\tau 1 \in {}^{\pi^*\tau} K^*(S)$ is unambiguously defined. If π is K-oriented, we obtain an index class $\pi_!{}^\tau 1 \in {}^\tau K^d(B)$. On a fibre S_b, $\pi^*\tau = 0$ as a cocycle, allowing one

[11] Its meaning in twisted K-theory will be discussed in [FHT], in connection with the Frobenius algebra structure.

to identify ${}^{\pi^*\tau}K^*(S_b)$ with $K^*(S_b)$. However, the new identification differs from the old one, over all of S, where $\pi^*\tau = \delta\omega$; so the restriction of ${}^{\tau}1$ corresponds to a line bundle $\mathcal{O}(\omega)$ over S_b with $c_1 = \omega$. The fibre of $\pi_!{}^{\tau}1$ at b is then the index of $\mathcal{O}(\omega)$ over S_b.

We apply this to our situation, where the index (7.5) should be captured by the map from the manifold X' of (7.2) to a point, in LG-equivariant K-theory — if the latter existed. Take for B the *classifying stack* BLG, and for S the *quotient stack* X'/LG. As a real manifold, X' is the moduli space of flat G-connections on Σ°, trivialized on the boundary; so X'/LG is equivalent to the quotient stack $G^{\times 2g}/G$. BLG has the homotopy type of G/G, and in these identifications, the projection $X' \to *$ becomes our map $\Pi: G^{\times 2g}/G \to G/G$.

Twistings are required, since the LG-action is projective on H^0 and on the line bundle. In §5, we identified ${}^{h+c}K_G^d(G)$ with the Verlinde algebra $V(h)$, which is where our analytic index lives. This leads us to the push-forward $\Pi_!: {}^{h+c}K_G^0(G^{2g}) \to {}^{h+c}K_G^0(G)$. We can reconcile this shifted twisting with the level h in (7.5) by reinterpreting the left-hand side there as the *Dirac index* of $\mathcal{O}(h+c)$; the c-shift is the projective cocycle of the LG-action on spinors on X. (This is one of the ways in which $\mathcal{O}(-2c)$ behaves like the canonical bundle of X). Thus, (7.5) and (7.6) express the equality of the analytic (algebraic, really) and topological indices.

We can summarize and clarify our discussion by introducing the space $\mathcal{A}$ of G-connections on $\partial\Sigma^\circ$. The stack $\mathcal{A}/LG$ is equivalent to G/G; so ${}^{h+c}K_G^d(G) = {}^{h+c}K_{LG}^d(\mathcal{A})$; moreover, the boundary restriction $X'/LG \to \mathcal{A}/LG$ is exactly Π. We are then saying that the LG-equivariant analytic index over X', rigorously defined in the algebraic model, is computed topologically, by factoring the push-forward to a point into the rigorously defined $\Pi_!: {}^{h+c}K_{LG}^0(X') \to {}^{h+c}K_{LG}^0(\mathcal{A})$, and the LG-equivariant push-forward $\mathcal{A} \to *$; the latter, we interpret as the isomorphism ${}^{h+c}K_G^d(G) \cong V(h)$.

Appendix A. Gradings, or twistings by $H^1(\mathbb{Z}/2)$

We discuss here the changes to Theorem (2.4) in the presence of an additional K-theory twisting $\varepsilon \in H_G^1(X;\mathbb{Z}/2)$; the final result (A.11) is not altogether obvious. The ideas and definitions should be clear to readers familiar with the K-theory of *graded* C^* algebras, as in [B], §14; see also Remark (A.7). However, we discuss the case of groups in more detail, which we need in order to understand the sheaves ${}^{\varepsilon}\mathcal{K}^*$. In this section, G is any compact Lie group.

When X is a point

In this case, ε gives a homomorphism $\varepsilon: G \to \{\pm 1\}$, which we call a $\mathbb{Z}/2$-*grading* of G. The fibers $G^\pm$ of ε are conjugation-stable unions of components of G. A *graded representation* is a $\mathbb{Z}/2$-graded vector space with a linear action of G, where even elements preserve, and odd ones reverse the grading. We use the notation $M^+ \ominus M^-$, the superscript indicating the eigenvalue of the *degree operator* D; the reason for this (purely symbolic) "direct difference" notation will be clear below. A *super-symmetry* of such a representation is an odd automorphism, skew-commuting with G; representations which admit a super-symmetry are called *super-symmetric*.

(A.1.i) Definition. ${}^{\varepsilon,\tau}R_G = {}^{\varepsilon,\tau}K_G^0(*)$ *is the abelian group of finite-dimensional, τ-projective, graded representations, modulo supersymmetric ones.*

The sum of any graded representation with a degree-reversed copy of itself is super-symmetric; be-

cause of this, flipping the grading acts as a sign change on K^0, and so (A.1.i) defines an abelian group and not just a semi-group. *Restriction of graded representations,* from G to G^+, sends $M^+\ominus M^-$ to the virtual representation $M^+ - M^-$. Restriction has a right adjoint *graded induction* functor from ${}^{\tau}R_{G^+}$ to ${}^{\varepsilon,\tau}R_G$. Both are R_G-module maps. It is fairly easy to see that graded induction identifies ${}^{\varepsilon,\tau}R_G$ with the cokernel of the ordinary restriction map from ${}^{\tau}R_G$ to ${}^{\tau}R_{G^+}$, and graded restriction with the kernel of ordinary induction from ${}^{\tau}R_{G^+}$ to ${}^{\tau}R_G$. (This is further clarified by the exact sequences (A.3) below). The last description shows that ${}^{\varepsilon,\tau}R_G$ is torsion-free.

We define the graded K^1 by an implicit Thom isomorphism. The group $G\times\mathbb{Z}/2$ carries a $\{\pm1\}$-valued 2-cocycle κ, lifted, under $\varepsilon\times\mathrm{Id}$, from the Heisenberg extension of $\mathbb{Z}/2\times\mathbb{Z}/2$. Call ν the non-trivial grading on $\mathbb{Z}/2$, and also, somewhat abusively, its pull-back to $G\times\mathbb{Z}/2$.

(A.1.ii) Definition. ${}^{\varepsilon,\tau}K^1_G(*):={}^{\nu,\tau+\kappa}R_{G\times\mathbb{Z}/2}$.

This can be made more concrete. Denote by ${}^{\varepsilon}\mathbb{C}$ the sign representation of G. A graded, $(\tau+\kappa)$-projective representation $M^+\ominus M^-$ of $(G\times\mathbb{Z}/2,\nu)$ is determined, up to canonical isomorphism, by M^+: the isomorphism $S:M^-\to M^+\otimes{}^{\varepsilon}\mathbb{C}$, defined by $(1,-1)\in G\times\mathbb{Z}/2$, pins down the other factor. Supersymmetric representations are those for which M^+ is isomorphic to its ε-twist; these are precisely the representations induced from G^+. In particular, ${}^{\varepsilon,\tau}K^1_G(*)\cong{}^{\tau}R_G\big/\mathrm{Ind}{}^{\tau}R_{G^+}$.

(A.2) Remark. The graded C^* algebra version of the Thom isomorphism identifies ${}^{\varepsilon,\tau}K^1_G(*)$ with the ${}^{\tau}K^0$-group of the *graded product* of (G,ε) with the rank 1 Clifford algebra $\mathbf{C}_1$. This graded product is the graded convolution algebra ${}^{\tau+\kappa}C_*(G\times\mathbb{Z}/2)$, with *product grading* $\varepsilon\times\nu$: the cocycle κ stems from the anticommutation of odd elements. Now, the right-hand side in (A.1.ii) is the ${}^{\tau+\kappa}K^0$–group of the convolution algebra of $(G\times\mathbb{Z}/2,\nu)$, which may seem different at first. However, an obvious shearing map identifies the latter with $(G\times\mathbb{Z}/2,\varepsilon\times\nu)$, as graded groups; and the shearing can be lifted to $\mathbb{T}$-extensions. (Restricted to the diagonal subgroup $G^+\times\{1\}\cup G^-\times\{-1\}$, κ is not trivial as a $\mathbb{Z}/2$-cocycle; its class is the square of ε. However, it is trivialized, as a $\mathbb{T}$-valued extension, after a choice of $\sqrt{-1}$, and this allows us to lift the shearing automorphism).

Identifying the trivially graded group G with the subgroup of even elements in $G\times\mathbb{Z}/2$ defines graded restriction and induction maps, $\mathrm{Res}':{}^{\varepsilon,\tau}K^1_G(*)\to{}^{\tau}R_G$, $\mathrm{Ind}':{}^{\tau}R_G\to{}^{\varepsilon,\tau}K^1_G(*)$. In our concrete description of K^1, they send M^+ to $M^+-M^+\otimes{}^{\varepsilon}\mathbb{C}$ and M to $M^+:=M$. These maps assemble into two exact sequences relating the graded and ungraded K-theories:

$$\text{(A.3.i)}\qquad 0\to{}^{\varepsilon,\tau}K^1_G(*)\xrightarrow{\mathrm{Res}'}{}^{\tau}K^0_G(*)\xrightarrow{\mathrm{Res}}{}^{\tau}K^0_{G^+}(*)\xrightarrow{\mathrm{Ind}}{}^{\varepsilon,\tau}K^0_G(*)\to 0$$

$$\text{(A.3.ii)}\qquad 0\leftarrow{}^{\varepsilon,\tau}K^1_G(*)\xleftarrow{\mathrm{Ind}'}{}^{\tau}K^0_G(*)\xleftarrow{\mathrm{Ind}}{}^{\tau}K^0_{G^+}(*)\xleftarrow{\mathrm{Res}}{}^{\varepsilon,\tau}K^0_G(*)\leftarrow 0.$$

Recall from §2 that τ defines a $G_{\mathbb{C}}$-equivariant line bundle ${}^{\tau}\mathcal{C}h$ over $G_{\mathbb{C}}$, whose invariant sections are spanned by the characters of irreducible, τ-projective representations. *Anti-invariant* sections are those transforming under the character ε of $G_{\mathbb{C}}$; the *super-character* of a graded representation, $g\mapsto Tr(Dg)$, is an example. The super-character of a graded G-representation is supported on G^+, because odd group elements are off-diagonal in the $M^{\pm}$-decomposition, while that of a $(\nu,\tau+\kappa)$-twisted representation of $G\times\mathbb{Z}/2$ lives on $G^-_{\mathbb{C}}\times\{1\}$, because $Tr(Dg)=Tr\left(SDgS^{-1}\right)=-Tr(Dg)$, if $g\in G^+$. The following is clear from the exact sequences (A.3).

(A.4) Proposition. *The $\mathbb{C}R_G$-module ${}^{\varepsilon,\tau}K_G^0(*;\mathbb{C})$ is isomorphic to the space of anti-invariant algebraic sections of ${}^{\tau}\mathcal{C}h$ on $G^+_{\mathbb{C}}$, while ${}^{\varepsilon,\tau}K_G^1(*;\mathbb{C})$ is isomorphic to the space of invariant sections of ${}^{\tau}\mathcal{C}h$ over $G^-_{\mathbb{C}}$. Both isomorphisms are realized by the super-character.* ✠

(A.5) Remark. In terms of the "odd line" $\mathbb{C}^-$, ${}^{\varepsilon,\tau}R_G\,\mathbb{C}^-$ is the graded module of skew-invariant sections of ${}^{\tau}\mathcal{C}h\otimes\mathbb{C}^-$ over G. Over G^+, $\mathbb{C}^-$-valued functions are odd, so skew invariance means anti-invariance. Over G^-, the sections are even, and skew invariance means invariance.

The General Case

On a space X, $\varepsilon\in H^1_G(X;\mathbb{Z}/2)$ defines a real G-line bundle $\mathcal{R}$ over X, with unit interval bundle D.

(A.6) Definition. *The groups ${}^{\varepsilon,\tau}K_G^*(X)$ are the relative ${}^{\tau}K_G^{*-1}$-groups of $(D,\partial D)$.*

Remark. This is really a twisted Thom isomorphism theorem; we thank G. Segal for the suggestion of using it as a definitional shortcut.

(A.7) Remark. The boundary ∂D is a double G-cover $p\colon\tilde{X}\to X$, and we can give a C^*-friendly description of ${}^{\varepsilon,\tau}K_G^*(X)$ as the ${}^{\tau}K^{*+1}$-groups of the crossed product $G\ltimes C^*(\tilde{X})$, graded by the eigenvalues of the deck transformation. These K-groups do *not* usually have a naive description in terms of graded projective modules, as in the case of a group algebra (A.1); see [B] for the definitions.

Note the vanishing of the first and third Stiefel-Whitney classes of the bundle $\mathcal{R}\oplus\mathcal{R}$ over X; the Thom isomorphism allows us to identify the doubly ε-twisted K-groups ${}^{2\varepsilon,\tau}K_G^*$ with ${}^{\tau}K_G^*$, but a choice of sign is needed.

The various K-groups are related by two six-term exact sequences analogous to (A.3), involving the double cover $\tilde{X}$ (for notational convenience, we omit the twisting τ, which is present everywhere). The first one is the six-term sequence for D and its boundary; the second follows from the first, by replacing X by the pair $(D,\partial D)$ and using the vanishing of $p^*\varepsilon$ on $\tilde{X}$.

(A.8.i)
$$\begin{array}{ccccc} K_G^0(X) & \xrightarrow{p^*} & K_G^0(\tilde{X}) & \xrightarrow{{}^{\varepsilon}p_!} & {}^{\varepsilon}K_G^0(X) \\ \uparrow & & & & \downarrow \\ {}^{\varepsilon}K_G^1(X) & \xleftarrow{{}^{\varepsilon}p_!} & K_G^1(\tilde{X}) & \xleftarrow{p^*} & K_G^1(X) \end{array}$$

(A.8.ii)
$$\begin{array}{ccccc} K_G^0(X) & \xleftarrow{p_!} & K_G^0(\tilde{X}) & \xleftarrow{{}^{\varepsilon}p^*} & {}^{\varepsilon}K_G^0(X) \\ \downarrow & & & & \uparrow \\ {}^{\varepsilon}K_G^1(X) & \xrightarrow{{}^{\varepsilon}p^*} & K_G^1(\tilde{X}) & \xrightarrow{p_!} & K_G^1(X) \end{array}$$

Moreover, denoting by α the non-trivial deck transformation on $\tilde{X}$ and by ${}^{\varepsilon}\mathcal{L}$ the flat line bundle $\mathcal{R}\otimes\mathbb{C}$ on X, we have $p_!p^*=(1+{}^{\varepsilon}\mathcal{L})\otimes$, ${}^{\varepsilon}p_!{}^{\varepsilon}p^*=(1-{}^{\varepsilon}\mathcal{L})\otimes$, $p^*p_!=(1+\alpha^*)$, ${}^{\varepsilon}p^*{}^{\varepsilon}p_!=(1-\alpha^*)$. Noting that the operations α^*, resp. tensoring with ${}^{\varepsilon}\mathcal{L}$, both square to 1, we can decompose, modulo 2-torsion, all the K-groups in (A.8) according to the eigenvalues of these operations.

(A.9) Proposition. ${}^{\varepsilon,\tau}K_G^0(X;\mathbb{C})\cong{}^{\tau}K_G^0(\tilde{X};\mathbb{C})^-\oplus{}^{\tau}K_G^1(X;\mathbb{C})^-$, ${}^{\varepsilon,\tau}K_G^1(X;\mathbb{C})\cong{}^{\tau}K_G^1(\tilde{X};\mathbb{C})^-\oplus{}^{\tau}K_G^0(X;\mathbb{C})^-$

where the superscripts indicate the eigenvalue of α^**, resp.* ${}^{\varepsilon}\mathcal{L}\otimes$. ✠

Localisation in the graded case
Finally, we have the following H^1-twisted version of the results in §2.

(A.10) Proposition. *The restriction* ${}^{\varepsilon,\tau}K_G^*(X;\mathbb{C})^\wedge_g \to {}^{\varepsilon,\tau}K_Z^*(X^g;\mathbb{C})^\wedge_g$ *is an isomorphism.*

Proof. Same as for Prop. (2.6), by reduction to homogeneous spaces, in view of our description (A.4) of the graded equivariant *K*-theory of a point. ✠

An element $g\in G$ is called *even* or *odd*, over a component *Y* of its fixed-point set X^g, according to its action on the fibres of ${}^{\varepsilon}\mathcal{L}$; the former happens when *g* fixes $p^{-1}(Y)$, the latter when it interchanges the fibres over *Y*. Let ${}^{\varepsilon,\tau}\mathcal{L}(g) := {}^{\tau}\mathcal{L}(g)\otimes {}^{\varepsilon}\mathcal{L}$.

(A.11) Theorem. *If g is even over a component Y of* X^g, ${}^{\varepsilon,\tau}K_Z^*(Y;\mathbb{C})^\wedge_g \cong {}^{\tau}H_Z^*\big(Y;{}^{\varepsilon,\tau}\mathcal{L}(g)\big)$*; while, if g is odd over Y,* ${}^{\varepsilon,\tau}K_Z^*(Y;\mathbb{C})^\wedge_g \cong {}^{\tau}H_Z^{*+1}\big(Y;{}^{\tau}\mathcal{L}(g)\big)$.

Proof. When *g* is even over *Y*, the proof of (2.14) proceeds without change, with the additional H^1-twisting on both sides, and no change is needed in the proof of (2.15). We can also give a different argument, based on (A.8). Irrespective of the parity of *g*, we have

$$(A.12)\qquad {}^{\varepsilon,\tau}K_Z^*(Y;\mathbb{C})^\wedge_g \cong {}^{\tau}K_Z^*\big(p^{-1}(Y);\mathbb{C}\big)^{\wedge -}_g \oplus {}^{\tau}K_Z^{*+1}(Y;\mathbb{C})^{\wedge -}_g.$$

When *g* is even, (2.4) gives natural isomorphisms

$$(A.13)\qquad {}^{\tau}K_Z^*\big(p^{-1}(Y);\mathbb{C}\big)^\wedge_g \cong {}^{\tau}H_Z^*\big(p^{-1}(Y);p^{*\tau}\mathcal{L}(g)\big),\qquad {}^{\tau}K_Z^{*+1}(Y;\mathbb{C})^\wedge_g \cong {}^{\tau}H_Z^{*+1}\big(Y;{}^{\tau}\mathcal{L}(g)\big).$$

The (-1)-eigenspace for a^* in the first term is ${}^{\tau}H_Z^*\big(Y;{}^{\varepsilon,\tau}\mathcal{L}(g)\big)$, whereas tensoring with ${}^{\varepsilon}\mathcal{L}$ on the second term acts as the identity, because $ch^{\varepsilon}\mathcal{L}=1$ and *g* acts trivially on its fibres; so the second summand in (A.12) is nil in this case. This vanishing also follows from the surjectivity of the complexified maps $p_!$ in (A.8.ii), which is clear when they are identified, by (2.4), with the maps

$$(A.14)\qquad p_!: {}^{\tau}H_Z^*\big(p^{-1}(Y);p^{*\tau}\mathcal{L}(g)\big)\to {}^{\tau}H_Z^*\big(Y;{}^{\tau}\mathcal{L}(g)\big).$$

On the other hand, when *g* is odd, it acts freely on $p^{-1}(Y)$, so ${}^{\tau}K_Z^*(p^{-1}(Y);\mathbb{C})^\wedge_g = 0$, by (2.6); whence, using either one of the sequences (A.8), we get that ${}^{\varepsilon,\tau}K_Z^*(Y;\mathbb{C})^\wedge_g \cong {}^{\tau}K_Z^{*+1}(Y;\mathbb{C})^\wedge_g$, which is ${}^{\tau}H_Z^{*+1}\big(Y;{}^{\tau}\mathcal{L}(g)\big)$, as in Theorem (2.4). ✠

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References

[A] M.F. Atiyah: K-Theory Past and Present. Preprint, math.KT/0012213

[AMW] A. Alekseev, E.Meinrenken, C.T. Woodward: Formulas of Verlinde type for non-simply connected groups. Preprint, math.SG/0005047

[AS1] M.F. Atiyah and G.B. Segal: Equivariant *K*-Theory and Completion. J. Differential Geometry **3** (1969)

[AS2] M.F. Atiyah and G.B. Segal: On equivariant Euler characteristics. J. Geom. Phys. **6** (1989)

[B] B. Blackadar: K-theory of *C**-algebras. MSRI Publications **5**, Cambridge, 1998

[BBM] P. Baum, J.-L. Brylinski, and R. MacPherson: Cohomologie équivariante délocalisée. C. R. Acad. Sci. Paris Sér. I Math. **300** (1985)

[BCMMS] P. Bouwknegt, A.L. Carey, V. Mathai, M.K. Murray, and D. Stevenson: Twisted *K*-theory and *K*-theory of bundle gerbes. Preprint, hep-th/0106194

[BG] J. Block, and E. Getzler: Equivariant cyclic homology and equivariant differential forms. Ann. Sci. École Norm. Sup. (4) **27** (1994)

[BL1] A. Beauville, and Y. Laszlo: Conformal blocks and generalized theta functions. Comm. Math. Phys. **164**, 385–419 (1994)

[BL2] A. Beauville, and Y. Laszlo: The Picard group of the moduli of G -bundles on a curve. Compositio Math. **112**, 183–216 (1998)

[Bo] A. Borel: Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes. Tôhoku Math. J. **13**, 216–240 (1961)

[BZ] J.-L. Brylinski and B. Zhang: Equivariant *K*-theory of compact connected Lie groups. *K*-Theory **20**, 23–36 (2000)

[DK] P. Donovan, M. Karoubi: Graded Brauer groups and *K*–theory with local coefficients. Inst. Hautes Études Sci. Publ. Math. **38**, 5–25 (1970)

[DV] M. Duflo and M. Vergne: Cohomologie équivariante et descente. Astérisque **215** (1993)

[F1] D.S. Freed: The Verlinde algebra is twisted equivariant *K*-theory. Turkish J. Math. **25**, 159–167 (2001)

[F2] D.S. Freed: Twisted *K*-theory and Loop groups. Preprint, math.AT/0206???

[FHT] D.S. Freed, M.J. Hopkins, and C. Teleman: Twisted *K*-theory and the Verlinde algebra. In preparation.

[G] I. Grojnowski: Delocalized elliptic equivariant cohomology. Preprint, 1994

[H] R. Hartshorne: Algebraic Geometry. Springer, 1977

[KN] S. Kumar, M.S. Narasimhan: Picard group of the moduli space of *G*-bundles. Math. Annalen. 308, 155–173 (1997)

[LU] E. Lupercio and B. Uribe: Loop Groupoids, Gerbes, and Twisted Sectors on Orbifolds. Preprint, math.AT/0110207

[M] H. Matsumura: Commutative Ring Theory. Cambridge University Press, 1986

[Mc] J. McCleary: A User's Guide to Spectral Sequences. Cambridge University Press, 2001

[PS] A. Pressley and G.B. Segal: Loop Groups. Oxford, 1988

[R] J. Rosenberg: Homological invariants of extensions of *C**–algebras. Operator algebras and applications, Part 1 (Kingston, Ont., 1980), 35–75. Proc. Sympos. Pure Math. **38**, Amer. Math. Soc. 1982

[Ro] I. Rosu: Equivariant *K*-theory and equivariant cohomology. Preprint, math.AT/9912088

[S] G.B. Segal: Equivariant *K*-theory. Inst. Hautes Études Sci. Publ. Math. **34** (1968)

[T] C. Teleman: Borel-Weil-Bott theory on the moduli stack of G-bundles over a curve. Invent. Math. **134**, 1–57 (1988)

[V] C. Vafa: Modular invariance and discrete torsion on orbifolds. Nucl. Phys. **B273**, 592–600 (1986)

[VW] C. Vafa and E. Witten: On orbifolds with discrete torsion. J. Geom. Phys. **15** (1995)